Comparing *ZFC* and the *V*-logic multiverse using MAXIMIZE

Matteo de Ceglie
`decegliematteo@gmail.com`

BLAST 2021
Contributed Session 1

9^{th} June 2021

UNIVERSITÄT SALZBURG

Outline

1. Preliminaries
2. #-generation
3. *ZFC* *vs* the *V*-logic Multiverse
4. Concluding remarks

Outline

1 Preliminaries

2 #-generation

3 *ZFC* *vs* the *V*-logic Multiverse

4 Concluding remarks

Outline

Outline

Outline

The universism/multiversism debate

Universism

There exists only one set theoretic universe, V, that contains all the possible sets and thus cannot be expanded.

Multiversism

There are several set theoretic universes, each one instantiating a different axiomatization and concept of set, and they can all be further expanded.

The universism/multiversism debate

Universism

There exists only one set theoretic universe, V, that contains all the possible sets and thus cannot be expanded.

Multiversism

There are several set theoretic universes, each one instantiating a different axiomatization and concept of set, and they can all be further expanded.

The *V*-logic Multiverse

- The infinitary logic V-logic, based upon the language $\mathcal{L}_{\kappa^+,\omega}$, is the background logic;
- The multiverse is the collection of all the possible outer extensions of the ground universe V generated through set-generic, class-generic, hyperclass forcing, etc.;

MAS For any first-order φ with parameters from V, if the sentence of V-logic expressing "$\bar{W}$ is an outer model of V satisfying φ" is consistent in V-logic, then there is a universe W that is an outer model of V and that satisfies φ.

The V-logic Multiverse

- The infinitary logic V-logic, based upon the language $\mathcal{L}_{\kappa^+,\omega}$, is the background logic;
- The multiverse is the collection of all the possible outer extensions of the ground universe V generated through set-generic, class-generic, hyperclass forcing, etc.;

MAS For any first-order φ with parameters from V, if the sentence of V-logic expressing "$\bar{W}$ is an outer model of V satisfying φ" is consistent in V-logic, then there is a universe W that is an outer model of V and that satisfies φ.

The V-logic Multiverse

- The infinitary logic V-logic, based upon the language $\mathcal{L}_{\kappa^+,\omega}$, is the background logic;
- The multiverse is the collection of all the possible outer extensions of the ground universe V generated through set-generic, class-generic, hyperclass forcing, etc.;

MAS For any first-order φ with parameters from V, if the sentence of V-logic expressing "$\bar{W}$ is an outer model of V satisfying φ" is consistent in V-logic, then there is a universe W that is an outer model of V and that satisfies φ.

MAXIMIZE

Principle

When deciding which foundational framework to use, we should prefer the one that proves more isomorphism types.

- $ZFC + V = L$ *vs* $ZFC + \exists 0^{\#}$;
- $ZFC + \exists 0^{\#}$ proves the existence of a non-constructible set;
- In turn, this means that we can prove that $ZFC + \exists 0^{\#}$ proves an isomorphism type that cannot be realised by constructible set (i.e. in $ZFC + V = L$);
- Hence, $ZFC + \exists 0^{\#}$ *maximizes* over $ZFC + V = L$

MAXIMIZE

Principle

When deciding which foundational framework to use, we should prefer the one that proves more isomorphism types.

- $ZFC + V = L$ *vs* $ZFC + \exists 0^{\#}$;
- $ZFC + \exists 0^{\#}$ proves the existence of a non-constructible set;
- In turn, this means that we can prove that $ZFC + \exists 0^{\#}$ proves an isomorphism type that cannot be realised by constructible set (i.e. in $ZFC + V = L$);
- Hence, $ZFC + \exists 0^{\#}$ *maximizes* over $ZFC + V = L$.

MAXIMIZE

Principle

When deciding which foundational framework to use, we should prefer the one that proves more isomorphism types.

- $ZFC + V = L$ vs $ZFC + \exists 0^{\#}$;
- $ZFC + \exists 0^{\#}$ proves the existence of a non-constructible set;
- In turn, this means that we can prove that $ZFC + \exists 0^{\#}$ proves an isomorphism type that cannot be realised by constructible set (i.e. in $ZFC + V = L$);
- Hence, $ZFC + \exists 0^{\#}$ *maximizes* over $ZFC + V = L$.

MAXIMIZE

Principle

When deciding which foundational framework to use, we should prefer the one that proves more isomorphism types.

- $ZFC + V = L$ *vs* $ZFC + \exists 0^{\#}$;
- $ZFC + \exists 0^{\#}$ proves the existence of a non-constructible set;
- In turn, this means that we can prove that $ZFC + \exists 0^{\#}$ proves an isomorphism type that cannot be realised by constructible set (i.e. in $ZFC + V = L$);
- Hence, $ZFC + \exists 0^{\#}$ *maximizes* over $ZFC + V = L$.

MAXIMIZE

Principle

When deciding which foundational framework to use, we should prefer the one that proves more isomorphism types.

- $ZFC + V = L$ *vs* $ZFC + \exists 0^{\#}$;
- $ZFC + \exists 0^{\#}$ proves the existence of a non-constructible set;
- In turn, this means that we can prove that $ZFC + \exists 0^{\#}$ proves an isomorphism type that cannot be realised by constructible set (i.e. in $ZFC + V = L$);
- Hence, $ZFC + \exists 0^{\#}$ *maximizes* over $ZFC + V = L$.

Outline

Class-iterated sharps

Definition [Antos, Barton, Friedman, Honzik]

A class iterated sharp is a transitive structure $\mathfrak{N} = (N, U)$ such that:

- it is a model of ZFC^- (i.e. ZFC without the Power Set Axiom);
- It has critical point κ, where κ is the largest cardinal and it is strongly inaccessible;
- (N, U) is amenable (i.e. $x \cap U \in N$ for any $x \in N$);
- U is a normal measure on κ in (N,U);
- it is iterable in the sense that all successive ultrapower iterations along class well-orders are well-founded.

Class-iterated sharps

Definition [Antos, Barton, Friedman, Honzik]

A class iterated sharp is a transitive structure $\mathfrak{N} = (N, U)$ such that:

- it is a model of ZFC^- (i.e. ZFC without the Power Set Axiom);
- It has critical point κ, where κ is the largest cardinal and it is strongly inaccessible;
- (N, U) is amenable (i.e. $x \cap U \in N$ for any $x \in N$);
- U is a normal measure on κ in (N,U);
- it is iterable in the sense that all successive ultrapower iterations along class well-orders are well-founded.

UNIVERSITÄT WÜRZBURG

Class-iterated sharps

Definition [Antos, Barton, Friedman, Honzik]

A class iterated sharp is a transitive structure $\mathfrak{N} = (N, U)$ such that:

- it is a model of ZFC^- (i.e. ZFC without the Power Set Axiom);
- It has critical point κ, where κ is the largest cardinal and it is strongly inaccessible;
- (N, U) is amenable (i.e. $x \cap U \in N$ for any $x \in N$);
- U is a normal measure on κ in (N,U);
- it is iterable in the sense that all successive ultrapower iterations along class well-orders are well-founded.

Class-iterated sharps

Definition [Antos, Barton, Friedman, Honzik]

A class iterated sharp is a transitive structure $\mathfrak{N} = (N, U)$ such that:

- it is a model of ZFC^- (i.e. ZFC without the Power Set Axiom);
- It has critical point κ, where κ is the largest cardinal and it is strongly inaccessible;
- (N, U) is amenable (i.e. $x \cap U \in N$ for any $x \in N$);
- U is a normal measure on κ in (N,U);
- it is iterable in the sense that all successive ultrapower iterations along class well-orders are well-founded.

Class-iterated sharps

Definition [Antos, Barton, Friedman, Honzik]

A class iterated sharp is a transitive structure $\mathfrak{N} = (N, U)$ such that:

- it is a model of ZFC^- (i.e. ZFC without the Power Set Axiom);
- It has critical point κ, where κ is the largest cardinal and it is strongly inaccessible;
- (N, U) is amenable (i.e. $x \cap U \in N$ for any $x \in N$);
- U is a normal measure on κ in (N,U);
- it is iterable in the sense that all successive ultrapower iterations along class well-orders are well-founded.

Class iterably sharp generated model

Definition [Antos, Barton, Friedman, Honzik]

A transitive model $\mathfrak{M} = (M, \in)$ is *class iterably sharp generated* iff there is a class-iterable sharp (N, U) and an iteration $N_0 \rightarrow N_1 \rightarrow N_2 \rightarrow \ldots$ such that $M = \cup_{\beta \in On^{\mathfrak{M}}} V_{\kappa_\beta}^{N_\beta}$.

Class iterably sharp generated model

Definition [Antos, Barton, Friedman, Honzik]

A transitive model $\mathfrak{M} = (M, \in)$ is *class iterably sharp generated* iff there is a class-iterable sharp (N, U) and an iteration $N_0 \to N_1 \to N_2 \to \dots$ such that $M = \cup_{\beta \in On^{\mathfrak{M}}} V_{\kappa_\beta}^{N_\beta}$.

- Any satisfaction obtainable in height extensions of $\mathfrak{M}$ adding ordinals is already reflected to an initial segment of $\mathfrak{M}$;
- We are able to coalesce many reflection principles into a single property of a model;
- E.g. A model $\mathfrak{M}$ being class iterably sharp generated entails reflection from $\mathfrak{M}$ to initial segments of n^{th}-order logic for any n.

Class iterably sharp generated model

Definition [Antos, Barton, Friedman, Honzik]

A transitive model $\mathfrak{M} = (M, \in)$ is *class iterably sharp generated* iff there is a class-iterable sharp (N, U) and an iteration $N_0 \to N_1 \to N_2 \to \ldots$ such that $M = \cup_{\beta \in On^{\mathfrak{M}}} V_{\kappa_\beta}^{N_\beta}$.

- Any satisfaction obtainable in height extensions of $\mathfrak{M}$ adding ordinals is already reflected to an initial segment of $\mathfrak{M}$;
- We are able to coalesce many reflection principles into a single property of a model;
- E.g. A model $\mathfrak{M}$ being class iterably sharp generated entails reflection from $\mathfrak{M}$ to initial segments of n^{th}-order logic for any n.

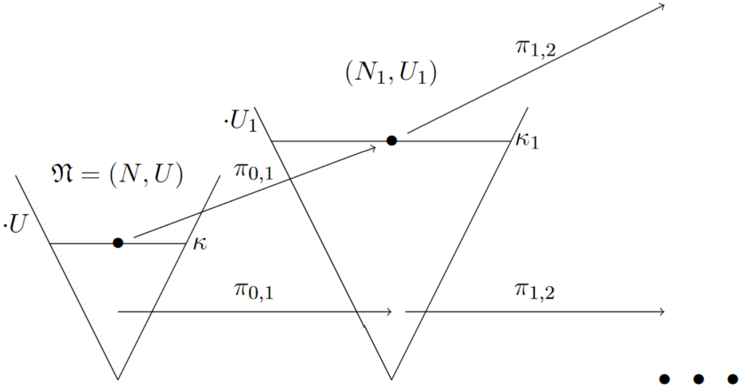
Class iterably sharp generated model

Definition [Antos, Barton, Friedman, Honzik]

A transitive model $\mathfrak{M} = (M, \in)$ is *class iterably sharp generated* iff there is a class-iterable sharp (N, U) and an iteration $N_0 \to N_1 \to N_2 \to \dots$ such that $M = \cup_{\beta \in On^{\mathfrak{M}}} V_{\kappa_\beta}^{N_\beta}$.

- Any satisfaction obtainable in height extensions of $\mathfrak{M}$ adding ordinals is already reflected to an initial segment of $\mathfrak{M}$;
- We are able to coalesce many reflection principles into a single property of a model;
- E.g. A model $\mathfrak{M}$ being class iterably sharp generated entails reflection from $\mathfrak{M}$ to initial segments of n^{th}-order logic for any n.

A useful picture



Outline

1 Preliminaries

2 #-generation

3 *ZFC* *vs* the *V*-logic Multiverse

4 Concluding remarks

The main claim

Claim

The V-logic multiverse (intended as $ZFC + LCs + MAS$) *maximizes over* the single universe framework (intended as $ZFC + LCs$).

- In the V-logic Multiverse we can find an object that it is not present in $ZFC + LCs$;
- Moreover, this object realises a new isomorphism type, that it is not available in $ZFC + LCs$.

The main claim

Claim

The V-logic multiverse (intended as $ZFC + LCs + MAS$) *maximizes over* the single universe framework (intended as $ZFC + LCs$).

- In the V-logic Multiverse we can find an object that it is not present in $ZFC + LCs$;
- Moreover, this object realises a new isomorphism type, that it is not available in $ZFC + LCs$.

The main claim

Claim

The V-logic multiverse (intended as $ZFC + LCs + MAS$) *maximizes over* the single universe framework (intended as $ZFC + LCs$).

- In the V-logic Multiverse we can find an object that it is not present in $ZFC + LCs$;
- Moreover, this object realises a new isomorphism type, that it is not available in $ZFC + LCs$.

Sketch of the argument

- In the V-logic Multiverse, there are proper, uncountable, generic extensions of V;
- If there are generic extensions of V, then there are iterable class sharps for V;
- Thus, V is class iterably sharp generated in the V-logic multiverse;
- In turn, this realises a new isomorphism type, between the models generated from class iterated sharps;
- We cannot have any of this in $ZFC + LCs$, since in that case we would be able to prove the existence of a cardinal that is both regular and singular.

Sketch of the argument

- In the V-logic Multiverse, there are proper, uncountable, generic extensions of V;
- If there are generic extensions of V, then there are iterable class sharps for V;
- Thus, V is class iterably sharp generated in the V-logic multiverse;
- In turn, this realises a new isomorphism type, between the models generated from class iterated sharps;
- We cannot have any of this in $ZFC + LCs$, since in that case we would be able to prove the existence of a cardinal that is both regular and singular.

Sketch of the argument

- In the V-logic Multiverse, there are proper, uncountable, generic extensions of V;
- If there are generic extensions of V, then there are iterable class sharps for V;
- Thus, V is class iterably sharp generated in the V-logic multiverse;
- In turn, this realises a new isomorphism type, between the models generated from class iterated sharps;
- We cannot have any of this in $ZFC + LCs$, since in that case we would be able to prove the existence of a cardinal that is both regular and singular.

Sketch of the argument

- In the V-logic Multiverse, there are proper, uncountable, generic extensions of V;
- If there are generic extensions of V, then there are iterable class sharps for V;
- Thus, V is class iterably sharp generated in the V-logic multiverse;
- In turn, this realises a new isomorphism type, between the models generated from class iterated sharps;
- We cannot have any of this in $ZFC + LCs$, since in that case we would be able to prove the existence of a cardinal that is both regular and singular.

Sketch of the argument

- In the V-logic Multiverse, there are proper, uncountable, generic extensions of V;
- If there are generic extensions of V, then there are iterable class sharps for V;
- Thus, V is class iterably sharp generated in the V-logic multiverse;
- In turn, this realises a new isomorphism type, between the models generated from class iterated sharps;
- We cannot have any of this in $ZFC + LCs$, since in that case we would be able to prove the existence of a cardinal that is both regular and singular.

Outline

The way forward

- Is this argument safe from a devil's advocate?
  - $ZFC + V = L$ implies that no transitive model of $ZFC + \exists 0^{\#}$ can contain any uncountable ordinal;
  - No countable ordinal can be isomorphic to an uncountable ordinal;
  - Hence, $ZFC + V = L$ realises an isomorphism type that cannot be found in $ZFC + \exists 0^{\#}$, thus it maximizes over it.
- Making this argument more formal:
  - Moving from isomorphism types to the notion of interpretation;
  - Proving that $ZFC + LCs$ is restrictive compared to $ZFC + LCs + MAS$;

The way forward

- Is this argument safe from a devil's advocate?
  - $ZFC + V = L$ implies that no transitive model of $ZFC + \exists 0^{\#}$ can contain any uncountable ordinal;
  - No countable ordinal can be isomorphic to an uncountable ordinal;
  - Hence, $ZFC + V = L$ realises an isomorphism type that cannot be found in $ZFC + \exists 0^{\#}$, thus it maximizes over it.
- Making this argument more formal:
  - Moving from isomorphism types to the notion of interpretation;
  - Proving that $ZFC + LCs$ is restrictive compared to $ZFC + LCs + MAS$;

The way forward

- Is this argument safe from a devil's advocate?
  - $ZFC + V = L$ implies that no transitive model of $ZFC + \exists 0^{\#}$ can contain any uncountable ordinal;
  - No countable ordinal can be isomorphic to an uncountable ordinal;
  - Hence, $ZFC + V = L$ realises an isomorphism type that cannot be found in $ZFC + \exists 0^{\#}$, thus it maximizes over it.
- Making this argument more formal:
  - Moving from isomorphism types to the notion of interpretation;
  - Proving that $ZFC + LCs$ is restrictive compared to $ZFC + LCs + MAS$;

UNIVERSITÄT SALZBURG

The way forward

- Is this argument safe from a devil's advocate?
  - $ZFC + V = L$ implies that no transitive model of $ZFC + \exists 0^{\#}$ can contain any uncountable ordinal;
  - No countable ordinal can be isomorphic to an uncountable ordinal;
  - Hence, $ZFC + V = L$ realises an isomorphism type that cannot be found in $ZFC + \exists 0^{\#}$, thus it maximizes over it.
- Making this argument more formal:
  - Moving from isomorphism types to the notion of interpretation;
  - Proving that $ZFC + LCs$ is restrictive compared to $ZFC + LCs + MAS$.

The way forward

- Is this argument safe from a devil's advocate?
  - $ZFC + V = L$ implies that no transitive model of $ZFC + \exists 0^{\#}$ can contain any uncountable ordinal;
  - No countable ordinal can be isomorphic to an uncountable ordinal;
  - Hence, $ZFC + V = L$ realises an isomorphism type that cannot be found in $ZFC + \exists 0^{\#}$, thus it maximizes over it.
- Making this argument more formal:
  - Moving from isomorphism types to the notion of interpretation;
  - Proving that $ZFC + LCs$ is *restrictive* compared to $ZFC + LCs + MAS$;

UNIVERSITÄT SALZBURG

The way forward

- Is this argument safe from a devil's advocate?
  - $ZFC + V = L$ implies that no transitive model of $ZFC + \exists 0^{\#}$ can contain any uncountable ordinal;
  - No countable ordinal can be isomorphic to an uncountable ordinal;
  - Hence, $ZFC + V = L$ realises an isomorphism type that cannot be found in $ZFC + \exists 0^{\#}$, thus it maximizes over it.
- Making this argument more formal:
  - Moving from isomorphism types to the notion of interpretation;
  - Proving that $ZFC + LCs$ is *restrictive* compared to $ZFC + LCs + MAS$;

The way forward

- Is this argument safe from a devil's advocate?
  - $ZFC + V = L$ implies that no transitive model of $ZFC + \exists 0^{\#}$ can contain any uncountable ordinal;
  - No countable ordinal can be isomorphic to an uncountable ordinal;
  - Hence, $ZFC + V = L$ realises an isomorphism type that cannot be found in $ZFC + \exists 0^{\#}$, thus it maximizes over it.
- Making this argument more formal:
  - Moving from isomorphism types to the notion of interpretation;
  - Proving that $ZFC + LCs$ is *restrictive* compared to $ZFC + LCs + MAS$;

Thank you very much for your attention!